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Self-avoiding surfaces

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Abstract. The set of self-avoiding random surfaces $\mathcal{S}_n(h)$ with n plaquettes and h boundary components is considered. The concatenation of surfaces in $\mathcal{S}_n(h)$ and new constructions which either increase or decrease the number of boundary components of a surface are studied. These constructions make it possible to prove the existence of growth constants, β_h, for the cardinality $s_n(h)$ of $\mathcal{S}_n(h)$ for each $h \geqslant 1$ in two dimensions and $h \geqslant 0$ in $d \geqslant 3$ dimensions. We prove that $\beta_h = \beta_1$ for all $h \geqslant 1$ in $d \geqslant 2$ dimensions. In addition, we prove that $\beta_0 < \beta_1$ in $d \geqslant 3$ dimensions and that in two dimensions $\beta_1 < \beta$, where β is the growth constant of the set $\mathcal{S}_n$, the set of all self-avoiding surfaces in two dimensions. Finally, by postulating the existence of a critical exponent ϕ_h for each set $\mathcal{S}_n(h)$, by assuming that $s_n(h) \sim n^{-\phi_h}\beta_h^n$, we derive bounds on ϕ_h from the constructions defined on the surface in $\mathcal{S}_n(h)$.

1. Introduction

Random surfaces have attracted a great deal of attention in recent years and have been proposed as models of a wide variety of different physical phenomena. These range from quantum field theory to solid state physics. In quantum field theory surfaces appear in the random surface representation of lattice gauge theories [1, 2], where it is hoped to provide an understanding of the confinement properties of QCD. The discrete version of the Polyakov string and the worldsheet action of the three-dimensional Ising model are also modelled by random surfaces [3]. A good review can be found in [4].

In condensed matter physics many problems involving interfaces and domain walls can be couched in terms of random surfaces [5-7], and self-avoiding surfaces have been used as models of membrane-like polymer networks [5, 8].

Glaus has recently studied two different kinds of self-avoiding surfaces in the simple cubic lattice in three dimensions [9]. These are surfaces homeomorphic to a sphere and a disc respectively. He presented numerical evidence that these surfaces are in the same universality class as branched polymers (or lattice animals) by estimating the value of an appropriate critical exponent (see also [10]).

Recently the importance of cycles in the statistics of lattice animals has been studied by numerical [11, 12] and analytical approaches [13, 14]. It has been shown that the critical exponent governing the number of animals with fixed cyclomatic index (c) is a function of c. (The cyclomatic index is the maximum number of edges which can be removed without disconnecting the animal.) In this paper we investigate a class of self-avoiding surfaces and show that there are strong similarities between the behaviour

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of surfaces and the behaviour of animals. In particular we consider surfaces with a fixed number of boundary components and present evidence that the boundary components in surfaces play a similar role to cycles in animals. Note that the number (h) of boundary components is related to the genus (g) of the surface and the rank (ρ) of its first homology group by the expression [15]

$$\rho = h + 2g - 1 \tag{1.1}$$

for $h \geqslant 1$.

Related to this work are the results of Durhuus *et al* [16, 17] who studied self-avoiding surfaces homeomorphic to a disc with fixed boundary γ. They proved that the surfaces grow exponentially fast with area and that the growth constant is independent of the boundary γ.

We shall be concerned with surfaces in the d-dimensional hypercubic lattice $\mathscr{Z}^d$, where $d \geqslant 2$. A self-avoiding surface is made up of elementary unit squares, or *plaquettes*. A plaquette is the interior and boundary of a unit square whose vertices are in the lattice $\mathscr{Z}^d$. We say that two plaquettes are *joined* if they share a common edge and that two plaquettes are *connected* if they are elements in a sequence of plaquettes such that neighbouring pairs of elements are joined. A *surface* is a collection of plaquettes such that each pair of plaquettes in the surface is connected. A surface is *self-avoiding* if each edge of a plaquette in the surface is incident on either one or two plaquettes. Edges incident on only one plaquette form part of the boundary of the surface, which may consist of several disjoint components. For instance, in three dimensions a surface homeomorphic to a disc has a single boundary component as does a surface homeomorphic to a Möbius strip. The boundary components can be linked and a single boundary component can be knotted, as in the Seifert surface of a trefoil.

Define $\mathscr{S}_n(h)$ to be the set of all self-avoiding surfaces in $\mathscr{Z}^d$ with h disjoint boundary components and consisting of n plaquettes. Two surfaces are regarded as distinct if they cannot be superimposed by translation. We include all surfaces, irrespective of genus, and irrespective of orientability. We note that if $h = 0$ then the surfaces are closed. Let $s_n(h)$ be the cardinality of $\mathscr{S}_n(h)$. Define $\mathscr{S}_n$ to be the union of all the sets $\mathscr{S}_n(h)$ and let s_n be the cardinality of $\mathscr{S}_n$.

Any self-avoiding surface $\sigma_n(h) \in \mathscr{S}_n(h)$ has a vertex set $\mathscr{V}$, an edge set $\mathscr{E}$ and a plaquette set $\mathscr{P}$. Let $\{e_i\}_{i=1}^d$ be the set of d orthogonal unit vectors in $\mathscr{Z}^d$. Then any vertex $v \in \mathscr{V}$ can be represented by a d-tuple $(v_1, v_2, \ldots, v_d) \in \mathscr{V} \subset \mathscr{Z}^d$. An edge in $\mathscr{E}$ can be represented by a pair (v, e_i) where v and $v + e_i$ are in $\mathscr{V}$ and are the endpoints of the edge. Similarly, a plaquette in $\mathscr{P}$ can be represented by a triple (v, e_i, e_j) where $e_i \cdot e_j = 0$ and v, $v + e_i$, $v + e_j$ and $v + e_i + e_j$ are elements of $\mathscr{V}$.

This paper is organised in the following way. In § 2 we prove that s_n is exponentially bounded from above (that is, there exists a positive number $K < \infty$ such that $n^{-1} \log s_n \leqslant \log K$ for all n). Consequently, $s_n(h)$ is also exponentially bounded from above. The concatenation of surfaces in the sets $\mathscr{S}_n(h)$ and $\mathscr{S}_n$ is considered in $d \geqslant 2$ dimensions in § 3. An immediate result of this concatenation is that the following limits exist:

$$\lim_{n \to \infty} n^{-1} \log s_n(0) = \log \beta_0 \qquad \text{if } d \geqslant 3 \tag{1.2}$$

$$\lim_{n \to \infty} n^{-1} \log s_n = \log \beta \qquad \text{if } d \geqslant 2. \tag{1.3}$$

Constructions on the surfaces in $\mathscr{S}_n(h)$ which will increase or decrease the number of boundary components ('drilling' and 'stripping') are considered in § 4. These additional constructions, in conjunction with concatenation, allow us to prove that the limit

$$\lim_{n\to\infty} n^{-1}\log s_n(h) = \log \beta_h \qquad \text{if } d \geqslant 2 \tag{1.4}$$

exists for all values of $h \geqslant 1$ and that

$$\beta_1 = \beta_h \qquad \forall h \geqslant 1. \tag{1.5}$$

An upper bound on $s_n(0)$ and a lower bound on $s_n(1)$ derived in section 5 in $d \geqslant 3$ dimensions proves that

$$\beta_0 < \beta_1 \qquad d \geqslant 3 \tag{1.6}$$

In two dimensions a result from § 4 provides a proof that

$$\beta_1 < \beta \qquad d = 2. \tag{1.7}$$

At present we are unable to extend this result to $d \geqslant 3$ dimensions. In § 6 we assume that in $d \geqslant 2$ dimensions there exist constants C_h for all $h \geqslant 1$ such that

$$s_n(h) \sim C_h n^{-\phi_h} \beta_h^n \tag{1.8}$$

where ϕ_h is an exponent to be determined. The following bounds on ϕ_h can then be proven:

$$\begin{cases} \phi_h \geqslant \phi_{h+1} \\ \phi_1 - h \geqslant \phi_{h+1} \geqslant \phi_1 - \frac{3}{2}h \end{cases} \qquad \forall h \geqslant 1 \text{ and } d = 2 \tag{1.9}$$

and

$$\phi_h \geqslant \phi_{h+1} \geqslant \phi_1 - 2h \qquad \forall h \geqslant 1 \text{ and } d \geqslant 3. \tag{1.10}$$

2. Exponential bounds on s_n and $s_n(h)$

In this section we prove that s_n is exponentially bounded as $n \to \infty$. This result is an essential ingredient in the proofs that limits like (1.2), (1.3) and (1.4) exist. We generalise a method of Eden [18] and Klarner [19] (applied to n-ominoes [20]) to construct an upper bound on s_n. This procedure has the advantage that it leads to a sequence of improved bounds if the surfaces are viewed as a sequence of *twigs* (see [21] for details).

Theorem 2.1. There exists a real constant $0 < K < \infty$ such that in $d \geqslant 2$ dimensions

$$s_n \leqslant \binom{d}{2} K^{n-1}$$

and consequently

$$s_n(h) \leqslant \binom{d}{2} K^{n-1}$$

where

$$\log K = (4(2d-3)+1)\log(4(2d-3)+1) - (4(2d-3)-1)\log(4(2d-3)-1).$$

Proof. Let $\sigma_n \in \mathcal{S}_n$ be a self-avoiding surface with n plaquettes in d dimensions with a plaquette set $\mathcal{P}$. Each plaquette can be represented by a triple (v, e_i, e_j) and has centre coordinates $c = v + \frac{1}{2}e_i + \frac{1}{2}e_j$. Since σ_n is connected, each plaquette has at least one and at most four nearest neighbours, each of which may be in at most $(2d-3)$ different orientations with respect to the plaquette. Choose the plaquette with minimum centre coordinates by defining the successive sets C_i by

$$C_1 = \left\{ p \in \mathcal{P} \,|\, c(p) = \{c_1^m, c_2, \ldots, c_d\}, c_1^m = \min_{p \in \mathcal{P}} \{c_1\} \right\}$$

$$C_k = \left\{ p \in C_{k-1} \,|\, c(p) = \{c_1^m, \ldots, c_k^m, c_{k+1}, \ldots, c_d\}, c_k^m = \min_{p \in C_{k-1}} \{c_k\} \right\}.$$

Since σ_n is self-avoiding, there exists a least $l \leqslant d$ such that C_l has only one element. Denote this plaquette by 1. Suppose this plaquette is oriented in the (e_i, e_j) plane, $i<j$. Denote the nearest neighbours of 1 by $2, 3, \ldots$ by first labelling the plaquette incident on the edge in the e_i direction of 1 (if there is a plaquette incident on this edge). Proceed then to the e_j direction, the $-e_i$ and $-e_j$ directions, labelling plaquettes successively, if they are present. Then proceed to the plaquette labelled 2 and label its unlabelled nearest neighbours in the same manner as above. Repeating this process will label all the plaquettes with numbers 1 to n in a unique way.

The edge (say (v, e_i)) of a plaquette labelled p (say (v, e_i, e_j)) can now be represented by a sequence of $(2d-3)$ binary digits B_l, where $1 \leqslant l \leqslant (2d-3)$. If this edge is not incident on a neighbouring plaquette, or if it is incident on a plaquette with label $q<p$, then we put all the B_l equal to zero. If this edge is incident on a plaquette $(v, e_i, (\pm)e_k)$ with label $r>p$, then one of the digits B_l is put equal to one. The neighbouring plaquette may have any of $(2d-3)$ orientations with respect to p, since if it is in the $(e_i, -e_k)$ plane, then $1 \leqslant k \leqslant d$ and $k \neq i, j$, and if it is in the (e_i, e_k) plane, then $1 \leqslant k \leqslant d$ and $k \neq i$. Let the digits B_l correspond to these orientations with increasing k, alternating between positive and negative directions. If the plaquette r is in a particular orientation, then that digit corresponding to this orientation is put equal to one. The plaquette p has four edges, so we can represent it by a sequence of $4(2d-3)$ binary digits, a sequence of $(2d-3)$ for each of the edges. For the plaquette (v, e_i, e_j) with $i<j$, we can uniquely order the edges in the sequence (v, e_i), (v, e_j), $(v+e_j, e_i)$ and $(v+e_i, e_j)$, and the whole surface can then be represented by $4(2d-3)(n-1)$ binary digits (since the last labelled plaquette, n, has no nearest neighbours with labels bigger than n), of which precisely $(n-1)$ digits are 1 (this is because we only put a digit in a sequence representing a plaquette labelled p equal to one if it corresponds to a neighbouring plaquette with label bigger than p). The number of ways that $(n-1)$ 1s can be chosen from $4(2d-3)(n-1)$ digits is

$$\binom{4(2d-3)(n-1)}{(n-1)}.$$

This number is exponentially bounded from above in n. Hence, since the plaquette with label 1 can be in $\binom{d}{2}$ orientations,

$$s_n \leqslant \binom{d}{2} K^{n-1}$$

where K is given by [22]

$$K = (4(2d-3)+1)\log(4(2d-3)+1) - (4(2d-3)-1)\log(4(2d-3)-1).$$

3. Concatenation of self-avoiding surfaces

The concatenation of two self-avoiding surfaces $\sigma_n(h) \in \mathscr{S}_n(h)$ and $\sigma_m(g) \in \mathscr{S}_m(g)$ in $d \geq 2$ dimensions is not an obvious operation. In particular, it is possible that the 'rightmost piece' of $\sigma_n(h)$ is an edge, while the 'leftmost piece' of $\sigma_m(g)$ is a plaquette, and it is not immediately obvious how to concatenate these surfaces. Consider the vertex set $\mathscr{V}$ of $\sigma_n(h)$ and label all the vertices by an index j, $v^{(j)} \in \mathscr{V}$. Define the successive sets $\mathscr{V}_i^t$ and $\mathscr{V}_i^b$ by the recursive equations

$$\mathscr{V}_1^t = \left\{ v \in \mathscr{V} \mid v = (v_1^t, v_2^{(j)}, \ldots, v_d^{(j)}),\ v_1^t = \max_j \{v_1^{(j)}\} \right\} \tag{3.1}$$

$$\mathscr{V}_i^t = \left\{ v \in \mathscr{V}_{i-1}^t \mid v = (v_1^t, \ldots, v_i^t, v_{i+1}^{(j)}, \ldots, v_d^{(j)}),\ v_i^t = \max_j \{v_i^{(j)}\} \right\} \tag{3.2}$$

and

$$\mathscr{V}_1^b = \left\{ v \in \mathscr{V} \mid v = (v_1^b, v_2^{(j)}, \ldots, v_d^{(j)}),\ v_1^b = \min_j \{v_1^{(j)}\} \right\} \tag{3.3}$$

$$\mathscr{V}_i^b = \left\{ v \in \mathscr{V}_{i-1}^b \mid v = (v_1^b, \ldots, v_i^b, v_{i+1}^{(j)}, \ldots, v_d^{(j)}),\ v_i^b = \min_j \{v_i^{(j)}\} \right\}. \tag{3.4}$$

Since $\sigma_n(h)$ is self-avoiding, there exists a smallest $i \leq d$ such that $\mathscr{V}_i^t$ has only one element. This element is called the *top vertex* t of $\sigma_n(h)$. There also exists a smallest $k \leq d$ such that $\mathscr{V}_k^b$ has only one element. This element is called the *bottom vertex* b of $\sigma_n(h)$.

The *top edge* of $\sigma_n(h)$ is found by considering the edges connected to the top vertex t. Let these edges be $\{(t, -e_{i_l})\}_{l=1}^{j}$, where $2 \leq j \leq d$, and suppose without loss of generality that $i_1 < i_2 < \ldots < i_j$. Since $j \geq 2$ there exists an l such that $e_1 \cdot e_{i_l} = 0$; let k be the smallest i_l for which this is true. Then $(t, -e_k)$ is the top edge, and it is perpendicular to e_1. There is always a top edge in any $\sigma_n(h)$.

The *bottom edge* of $\sigma_n(h)$ is found by considering the edges connected to the bottom vertex b. Let these edges be $\{(b, e_{i_l})\}_{l=1}^{j}$ where $2 \leq j \leq d$ and suppose again that $i_1 < i_2 < \ldots < i_j$. Since $j \geq 2$ there exists a l such that $e_1 \cdot e_{i_l} = 0$; let k be the smallest i_l for which this is true. Then (b, e_k) is the bottom edge, perpendicular to e_1. There is always a bottom edge.

A plaquette (v, e_i, e_j) is *perpendicular* to e_k if $e_i \cdot e_k = e_j \cdot e_k = 0$. This definition leads to the following lemma.

Lemma 3.1. If the top edge $(t, -e_t)$ of a surface $\sigma_n(h)$ is incident on two plaquettes $(t, -e_t, -e_i)$ and $(t, -e_t, -e_j)$, then at least one of these plaquettes is perpendicular to e_1. Similarly, if the bottom edge of a surface $\sigma_n(h)$ is incident on two plaquettes, then at least one of these plaquettes is perpendicular to e_1.

Proof. Since $e_t \cdot e_1 = e_b \cdot e_1 = 0$ by definition of the top and bottom edges, we only need to show that either $e_i \cdot e_1 = 0$ or $e_j \cdot e_1 = 0$. Suppose, without loss of generality, that $i < j$. Suppose that $e_i \cdot e_1 = 1$, then $i = 1$ and $j \neq 1$; hence $e_j \cdot e_1 = 0$ and $(t, -e_t, -e_j)$ is perpendicular to e_1. If $e_i \cdot e_1 = 0$, then $(t, -e_t, -e_i)$ is perpendicular to e_1. The same arguments hold for the bottom edge.

If the top edge of $\sigma_n(h)$ is incident on two plaquettes $(t, -e_t, -e_i)$ and $(t, -e_t, -e_j)$, then by lemma 3.1 at least one of these plaquettes is perpendicular to e_1. The *top plaquette* of $\sigma_n(h)$ is found by choosing that plaquette perpendicular to e_1 with the smallest value of i or j. The *bottom plaquette* is found in the same way. If the top or bottom edge of the surface is incident on only one plaquette, then it is not possible to define a top or a bottom plaquette.

It is now easy to study the concatenation of surfaces in two dimensions. The surfaces $\sigma_n(h)$ and $\sigma_m(g)$ have top and bottom edges. In two dimensions concatention is a map from $\mathscr{S}_n(h)\times\mathscr{S}_m(g)$ to $\mathscr{S}_{n+m}(h+g-1)$ defined by identifying the top edge of $\sigma_n(h)$ with the bottom edge of $\sigma_m(g)$:

$$\sigma_n(h) \underset{2}{\oplus} \sigma_m(g) \mapsto \sigma_{n+m}(h+g-1). \tag{3.5}$$

In $d \geqslant 3$ dimensions concatenation is more complicated. There are three possibilities which must be considered.

(1) $\sigma_n(h)$ has a top plaquette and $\sigma_m(g)$ has a bottom plaquette.

(2) $\sigma_n(h)$ has a top edge while $\sigma_m(g)$ has a bottom plaquette (or vice versa).

(3) $\sigma_n(h)$ has a top edge and $\sigma_m(g)$ has a bottom edge.

Concatenation can be defined in various ways. We shall require it to have the following properties.

(i) It must be injective.

(ii) The number of boundary components must be additive under the construction.

(iii) A fixed number of plaquettes must be added in each case.

The details of the construction are straightforward but tedious and we give only an outline. Consider the possibilities (1) to (3) above in turn.

(1) Let the top plaquette of $\sigma_n(h)$ be $(t, -e_i, -e_j)$ and the bottom plaquette of $\sigma_m(g)$ be (b, e_k, e_l) (figure 1). These plaquettes may have different orientations so that it may be impossible to translate $\sigma_m(g)$ such that they coincide. Instead, we change the orientation of the top and bottom plaquettes by adding plaquettes to $\sigma_n(h)$ and $\sigma_m(g)$. In figure 1 we illustrate how the orientation of the bottom plaquette may be changed by the addition of the cube $(b, e_k, e_l, -e_j)$ to it. It can be shown that by adding three such cubes, we can define concatenation to have the properties set out above. Thus, counting the number of plaquettes added we find a map

$$\sigma_n(h)\oplus\sigma_m(g)\mapsto\sigma_{m+n+10}(h+g). \tag{3.6}$$

(2) Let the top edge of $\sigma_n(h)$ be $(t, -e_i)$ and the bottom edge of $\sigma_m(g)$ be (b, e_j, e_k). The top edge and bottom plaquette may have different orientations so that it may be

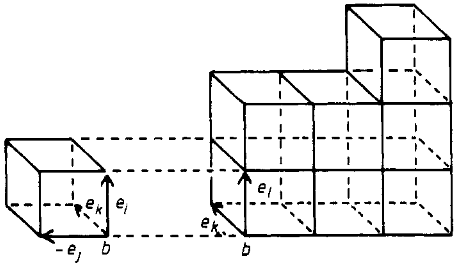

Figure 1. The concatenation of a cube $(b, -e_j, e_k, e_l)$ onto the bottom plaquette (b, e_k, e_l) by identifying their faces (b, e_k, e_l).

impossible to translate $\sigma_m(g)$ such that the top edge coincides with an edge of the bottom plaquette. The approach in this case is to construct a top plaquette for $\sigma_n(h)$ onto its top edge. This is accomplished by adding plaquettes to the top edge as shown in figure 2. We then add an additional set of plaquettes to take account of the possible orientations, as above. The outcome is again equation (3.6).

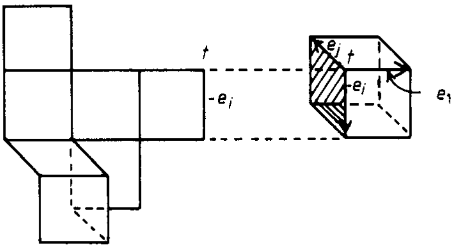

Figure 2. Concatenation on a top edge which is on a boundary component. The cube $(t, -e_i, e_j, e_1)^0$ with its $(t, -e_i, e_j)$ face deleted is concatenated onto the top edge $(t, -e_i)$ by identifying the edges $(t, -e_i)$.

(3) In this case we cannot simply identify the top and bottom edges since the number of boundary components do not then add. Consequently we add sets of plaquettes to construct top and bottom plaquettes as in figure 2. The outcome is again equation (3.6).

We have now considered all possible cases, and we have constructed the operation such that (3.5) holds in two dimensions while (3.6) is true in $d \geqslant 3$ dimensions. We can summarise the results in the following lemma.

Lemma 3.2. These exists a concatenation of self-avoiding surfaces in $d \geqslant 2$ dimensions which is a one-to-one map

$$\mathcal{S}_n(h) \underset{2}{\oplus} \mathcal{S}_m(g) \rightarrow \mathcal{S}_{n+m}(h+g-1) \qquad d=2$$

$$\mathcal{S}_n(h) \oplus \mathcal{S}_m(g) \rightarrow \mathcal{S}_{n+m+10}(h+g) \qquad d \geqslant 3$$

mapping pairs $(\sigma_n(h), \sigma_m(g)) \mapsto \sigma_{n+m+l}(h+g-k)$ where $l=0$ and $k=1$ in two dimensions and $l=10$ and $k=0$ in $d \geqslant 3$ dimensions.

The fact that concatenation is an injective map is easily seen. The concatenation construction does not destroy any of the edges of the two surfaces involved, and thus maps vertices onto vertices in a one-to-one fashion. From the definitions of the top edge and plaquette and from lemma 3.2 it is now easy to prove the following results:

Theorem 3.3. The cardinality of $\mathcal{S}_n(h)$, $s_n(h)$, obeys the following inequalities:

(i) $s_{n+1}(h) \geqslant s_n(h)$ $\quad \forall n, \forall h \geqslant 1$, and $d=2$

(ii) $s_{n+4}(h) \geqslant s_n(h)$ $\quad \forall n, \forall h \geqslant 0$, and $d \geqslant 3$

(iii) $s_n(h) s_m(g) \leqslant s_{n+m}(h+g-1)$ $\quad \forall n, m, \forall h, g \geqslant 1$, and $d=2$

(iv) $s_n(h) s_m(g) \leqslant s_{n+m+10}(h+g)$ $\quad \forall n, m, \forall h, g \geqslant 0$, and $d \geqslant 3$

(v) $s_n s_m \leqslant s_{n+m+l}$ $\quad \forall n, m$, and $d \geqslant 2$

where l is 0 in two dimensions and 10 in $d \geqslant 3$ dimensions.

Finally, we can prove the existence of some of the growth constants of sets of random surfaces. These results are stated together in the following theorem.

Theorem 3.4. These exist finite, positive real numbers β_0, β_1, β which depend on d such that

(i) $$\lim_{n\to\infty} n^{-1}\log s_n(1) = \sup_{n>0} n^{-1}\log s_n(1) = \log\beta_1 \qquad \text{if } d=2$$

(ii) $$\lim_{n\to\infty} n^{-1}\log s_n(0) = \log\beta_0 \text{ and } s_n(0)\leqslant \beta_0^{n+10} \qquad \text{if } d\geqslant 3$$

(iii) $$\lim_{n\to\infty} n^{-1}\log s_n = \log\beta$$

where $s_n \leqslant \beta^n$ in two dimensions and $s_n \leqslant \beta^{n+10}$ in $d\geqslant 3$ dimensions.

Proof. (i) Put $h=g=1$ in theorem 3.3 (iii). Then $s_n(1)s_m(1)\leqslant s_{n+m}(1)$. Since $s_n(1)$ is a sequence of positive numbers such that $s_n(1)^{1/n}$ is bounded above (theorem 2.1) satisfying this inequality, there exists a finite, positive constant β_1 such that (i) is satisfied [23].

(ii) Put $h=g=0$ in theorem 3.3 (iv). Then $s_n(0)$ is a sequence of positive numbers such that $s_n(0)^{1/n}$ is bounded above (theorem 2.1) and $s_n(0)s_m(0)\leqslant s_{n+m+10}(0)$. Thus there exists a finite, positive constant β_0 such that (ii) is true [24].

(iii) s_n is a sequence of positive numbers in $d\geqslant 2$ dimensions such that $s_n^{1/n}$ is bounded from above (theorem 2.1). Together with the results of concatenation and [23, 24], the theorem is proven.

4. Drilling and stripping

Concatenation provided a means whereby we could derive inequalities between the numbers $s_n(h)$ for different numbers of boundary components. In this section we investigate other constructions which will increase (drill) or decrease (strip) the number of boundary components in a given element of $\mathcal{S}_n$.

4.1. Drilling

Constructing new boundary components on a given random surface at an *a priori* location is a complicated procedure under the self-avoiding condition. In our efforts to construct ('drill') new boundaries we shall be successful only in two dimensions.

Lemma 4.1. Let $d=2$. Suppose that $\sigma_n(1)\in\mathcal{S}_n(1)$. Then it is possible to construct boundary components in at least $\lfloor \frac{n}{49}\rfloor$ locations on $\sigma_n(1)$. Furthermore, there exists a finite, positive constant C such that

$$\binom{\lfloor \frac{n}{49}\rfloor}{h} s_n(1)\leqslant C^h s_{n+48h}(1+h)$$

where $\lfloor x\rfloor$ is the largest integer less than or equal to x.

Proof. Let D be a square with $7\times 7=49$ plaquettes in the square lattice. $\sigma_n(1)$ can be completely covered by disjoint copies of D as illustrated in figure 3(a). Choose any D. If we can create a new boundary component here, then we have shown that we

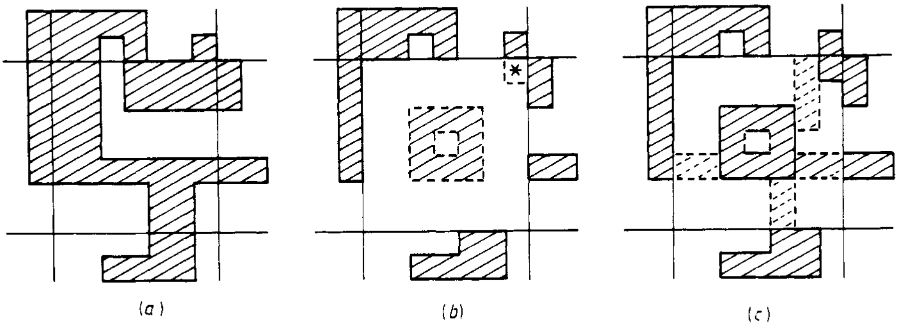

Figure 3. Drilling a hole in $\mathscr{S}_n(1)$ in two dimensions. In (a) a square of size 7×7 is identified and deleted. In (b) a plaquette is put back at the site marked $*$ to reconnect the two pieces which are now only touching at one vertex. The final step in (c) is to create a new boundary component in the manner indicated and to reconnect the surface by putting back plaquettes as shown in the figure.

can create a new boundary component in at least $\lfloor \frac{n}{49} \rfloor$ locations, since this is the minimum number of copies of D necessary to cover $\sigma_n(1)$. Since $\sigma_n(1)$ contains only one boundary component, D could not completely contain an entire boundary curve (unless it covers $\sigma_n(1)$ completely, which is the trivial case). Suppose that C' is the total number of surfaces which are identical outside D but are different from each other inside D. Then C' is bounded by the total number of ways that at most 49 plaquettes can be packed into D, say C''. To create a new boundary component, begin by deleting all the plaquettes of $\sigma_n(1)$ contained inside D (figure 3(b)). This deletion of plaquettes may create a hole in $\sigma_n(1)$, and then we are finished, having reduced the area of $\sigma_n(1)$ by 49 (this will only happen if D does not intersect any piece of the boundary component of $\sigma_n(1)$). In most cases, however, the situation will be like that in figure 3(b).

In general, deleting the square D will separate $\sigma_n(1)$ into a number of different pieces, at most 16, since this is the maximum number of times that the boundary component can cross the boundary of D. At the four corners of D there is the possibility that two of these pieces may share a vertex, as illustrated by the $*$ in figure 3(b). We deal with these cases first by putting a single plaquette at the $*$. This reconnects the two pieces. Once this is done, we may still be left with 14 disjoint pieces of $\sigma_n(1)$. The next step is to create a new boundary component. This is achieved by constructing a 3×3 square with its central plaquette missing at the centre of D, as illustrated in figure 3(c). The last step is to reconnect the different pieces of $\sigma_n(1)$. This is done by putting plaquettes back between the newly created 3×3 square and the disjoint pieces. At most three plaquettes are needed to reconnect each piece; this process is illustrated in figure 3(c). This process is not unique but, since D is finite, there is a maximum number of ways that it can be done. Absorb this number into the constant C''. In principle, the total number of plaquettes that we may have to put back is 48, stemming from the fact that D has area 49 but that we must leave a hole at its centre. The total area of $\sigma_n(1)$ may thus increase by as much as 48.

The construction described is a map

$$\mathscr{S}_n(1) \rightarrow \bigcup_{j=n-49}^{n+48} \mathscr{S}_j(2)$$

which is at most C'' onto 1. For the cardinality of the sets we thus find

$$s_n(1) \leqslant C'' \sum_{j=n-49}^{n+48} s_j(2) \leqslant 98C'' s_{n+48}(2)$$

where the second inequality follows directly from theorem 3.3(i). Put $C = 98C''$. To prove the lemma, choose h locations from $\lfloor \frac{n}{49} \rfloor$ ways to drill h new boundary components.

It is not possible to repeat the construction of lemma 4.1 in $d \geqslant 3$ dimensions. The fact that the surfaces are self-avoiding makes it impossible to cover surfaces in $d \geqslant 3$ dimensions with squares as was done in two dimensions. The only way which we can use to increase the number of boundary component of a surface $\sigma_n(h) \in \mathcal{S}_n(h)$ in $d \geqslant 3$ dimensions is by concatenation as described in theorem 3.3(iv) where one can put $g = 1$.

4.2. Stripping

The next construction to be defined on the set $\mathcal{S}_n$ will be designed to reduce the number of boundary components in any given random surface. The strategy is to remove a strip of plaquettes between one boundary component and another. We are able to prove a slightly stronger result in $d = 2$ dimensions than in $d \geqslant 3$ dimensions.

Lemma 4.2. Let $d = 2$. Suppose that $\sigma_n(h) \in \mathcal{S}_n(h)$ where $h \geqslant 2$. Then the number of boundary components in $\sigma_n(h)$ can be reduced by 1 by removing at most $\lceil \frac{1}{2}(\sqrt{2n+1} - 1) \rceil$ plaquettes, where $\lceil x \rceil$ is the smallest integer greater than or equal to x. Furthermore

$$s_n(h) \leqslant \left\lfloor \frac{n-6}{2} \right\rfloor (\lceil \tfrac{1}{2}(\sqrt{2n+1} - 1) \rceil - 1) s_n(h-1).$$

Proof. Consider first the case for $h = 2$. Consider the annulus A in figure 4 where the shaded strip is to be removed. If there are n plaquettes in the annulus, then a simple calculation shows that at most $\lceil \frac{1}{2}(\sqrt{2n+1} - 1) \rceil$ plaquettes will be removed when the

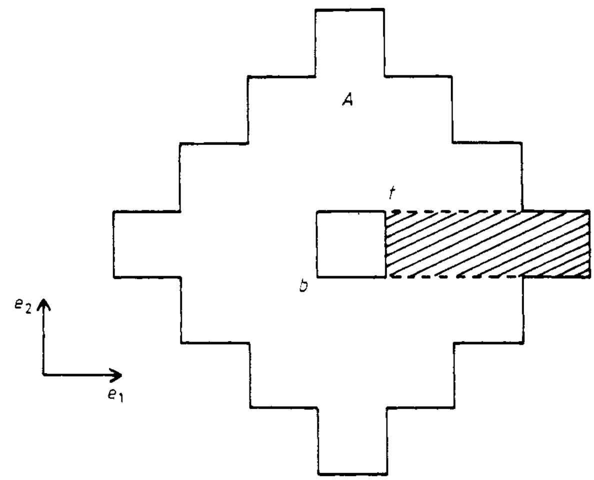

Figure 4. Cutting a strip of plaquettes (the shaded strip) from this surface will reduce the number of boundary components to one.

two boundary components are joined. The smaller boundary component of A has a top vertex t and a bottom vertex b as marked in figure 4. The situation in which the maximum number of plaquettes have to be removed, in order to join up two boundary components, is for surfaces such as A in figure 4. To see this, consider any B as in figure 5, also a surface with n plaquettes, but suppose that we have to remove at least $m > \lceil\frac{1}{2}(\sqrt{2n+1}-1)\rceil$ plaquettes to join the two boundary components. Like A, the smaller boundary component of B has a top vertex t' and bottom vertex b' in figure 5. Superimpose A and B by letting t and t' coincide. Then every plaquette of A with centre coordinates (as defined in theorem 2.1) such that the first component is strictly greater than the first component of t, or with first coordinate equal to and second coordinate strictly greater than the coordinates of $t-\frac{1}{2}(e_1+e_2)$, is covered by a plaquette of B (since $m > \lceil\frac{1}{2}(\sqrt{2n+1}-1)\rceil$), the shaded area in figure 5. In the same way, if we superimpose A and B by letting b and b' coincide, all the plaquettes of A with centre coordinates with first components strictly smaller than the first component of b, or with first components equal to and second coordinates strictly smaller than the coordinates of $b+\frac{1}{2}(e_1+e_2)$, are covered with plaquettes of B. So every plaquette of A can be covered by a plaquette of B. Since the inequality between m and the number of plaquettes of A to be removed is strict, there are some plaquettes of B that do not cover any of the plaquettes of A, so B must have more than n plaquettes. This is a contradiction, since we have assumed that B has only n plaquettes; hence $m \leqslant \lceil\frac{1}{2}(\sqrt{2n+1}-1)\rceil$. In the general case, with more than two boundary components, we find the minimum number of plaquettes to be removed by superimposing figure 4 onto the surface. The same arguments as above then hold. It is easily checked that there is precisely one special case. A hole the size of a single plaquette having four plaquettes incident on it such that deleting any will disconnect the surface. In this case we delete the plaquette incident on the top vertex of the hole and fill the hole in.

The construction defined so far maps

$$\mathscr{S}_n(h) \to \bigcup_{j=n-z}^{n} \mathscr{S}_j(h-1)$$

where $z = \lceil\frac{1}{2}(\sqrt{2n+1}-1)\rceil$, but not one-to-one. The worst case is illustrated in figure 6, where the shaded plaquette can be in $\lfloor\frac{n-6}{2}\rfloor$ positions; its removal will reduce the

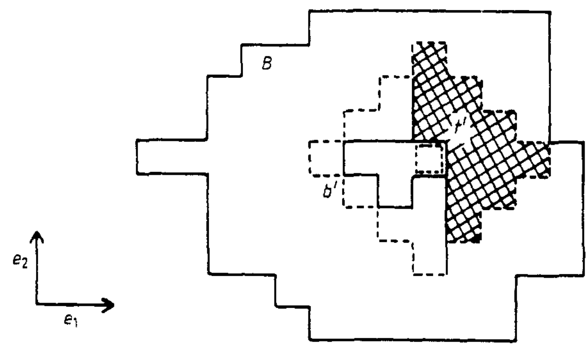

Figure 5. To find the least number of plaquettes to be removed in order to reduce the number of boundary components, figure 4 is superimposed on figure 5 by letting t and t' coincide. The shaded plaquettes are those plaquettes of figure 4 that are covered by a plaquette of B. The process is then repeated by letting b and b' coincide.

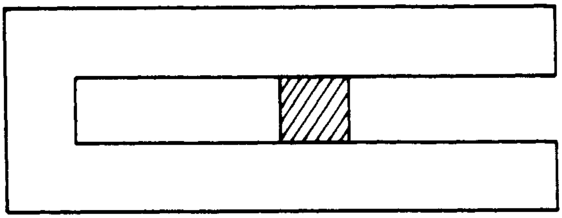

Figure 6. The shaded plaquette can be removed from $\lfloor\frac{n-6}{2}\rfloor$ places and would give the same resulting surface.

number of boundaries by 1, each time producing the same result. Hence

$$s_n(h) \leq \left\lfloor \frac{n-6}{2} \right\rfloor \sum_{j=n-z}^{n} s_j(h-1) \leq \left\lfloor \frac{n-6}{2} \right\rfloor (\lceil \tfrac{1}{2}(\sqrt{2n+1}-1)\rceil - 1) s_n(h-1)$$

where we have used theorem 3.3(i) in the second inequality.

Corollary 4.3. Let $d=2$. Suppose that $\sigma_n(h) \in \mathcal{S}_n(h)$ where $h \geq 2$. Then we can reduce the number of boundary components in $\sigma_n(h)$ by $g \leq h-1$ to find

$$\binom{h}{g} s_n(h) \leq \left(\left\lfloor \frac{n-6}{2} \right\rfloor (\lceil \tfrac{1}{2}(\sqrt{2n+1}-1)\rceil - 1) \right)^g s_n(h-g).$$

Proof. Choose g boundaries from h and apply lemma 4.2.

A similar proof can be constructed in $d \geq 3$ dimensions. The result is stated in the following lemma.

Lemma 4.4. Let $d \geq 3$. Suppose that $\sigma_n(h) \in \mathcal{S}_n(h)$ where $h \geq 2$. Then the number of boundary components in $\sigma_n(h)$ can be reduced by one by removing a strip of at most $\lceil \frac{n}{4} \rceil$ plaquettes. Furthermore

$$s_n(h) \leq (n-3) \left(\left\lceil \frac{n}{4} \right\rceil - 1 \right) s_{n+i_n}(h-1)$$

where i_n is a number such that $-1 \leq i_n \leq 2$.

Proof. As in two dimensions, consider the worst case first. Figure 7 is a hollow tube with two boundaries at its ends. To reduce the number of boundary components to one, it is necessary to remove $\lceil \frac{n}{4} \rceil$ plaquettes (the shaded stip in figure 7). This construction joins up the two boundary components. To see that all other cases are worse, consider any $\sigma_n(2)$ and suppose that we have to remove at least $m > \lceil \frac{n}{4} \rceil$ plaquettes to join the two boundaries. Shade this strip of plaquettes as in figure 7. Then there are two strips of plaquettes adjacent to the shaded strip, one on each side, each containing at least m plaquettes which can be removed instead to join the

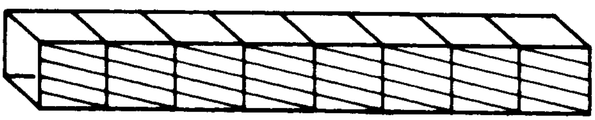

Figure 7. In this figure we have to remove $\lceil \frac{n}{4} \rceil$ plaquettes before the number of boundary components will be reduced to one. The boundary components are at the endpoints of the tube.

boundaries. Shade these strips too. Lastly, at least one of the two strips last shaded contains a fourth adjacent strip of at least m plaquettes which can be removed to join the two boundaries (this is because $\mathscr{Z}^d$ has girth 4). So $\sigma_n(2)$ has a surface area of at least $4m > n$. But this is a contradiction, so $\lceil \frac{n}{4} \rceil$ is the maximum number of plaquettes to be removed. If a surface has more than two boundary components, then the strips are removed in the same way, but not necessarily between the same two boundaries, and the outcome is still a contradiction.

This operation maps

$$\mathscr{S}_n(h) \rightarrow \bigcup_{j=n-\lceil \frac{n}{4} \rceil}^{n-1} \mathscr{S}_j(h-1)$$

but the mapping is not one-to-one. The worst case is illustrated in figure 8. Here $(n-3)$ elements of $\mathscr{S}_n(h)$ will map onto a single element of $\mathscr{S}_{n-1}(h-1)$ if the shaded plaquette is removed. Hence

$$s_n(h) \leqslant (n-3) \sum_{j=\lceil \frac{3n}{4} \rceil}^{n-1} s_j(h-1).$$

From theorem 3.3(ii) we have the result that $s_n(h) \leqslant s_{n+4}(h)$. So to each j in $s_j(h-1)$ in the above equation, add $n + i_j - j$; i_j chosen such that $n + i_j - j$ is a multiple of four, while applying the inequality in theorem 3.3(ii). Certainly one can choose $-1 \leqslant i_j \leqslant 2$. Thus

$$s_n(h) \leqslant (n-3) \sum_{j=\lceil \frac{3n}{4} \rceil}^{n-1} s_{n+i_j}(h-1)$$

since each $s_j(h-1) \leqslant s_{n+i_j}(h-1)$ for $-1 \leqslant i_j \leqslant 2$. For each n define i_n by

$$s_{n+i_n}(h-1) = \max_{-1 \leqslant i \leqslant 2} s_{n+i}(h-1)$$

and substitute into the above equation. Performing the sum over j then proves the theorem.

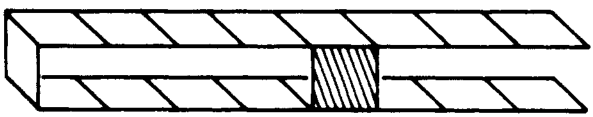

Figure 8. The shaded plaquette can be removed from $(n-1)$ positions to give the same outcome.

Corollary 4.5. Let $d \geqslant 3$. Suppose that $\sigma_n(h) \in \mathscr{S}_n(h)$ where $h \geqslant 2$. Then we can reduce the number of boundary components in $\sigma_n(h)$ by $g \leqslant h-1$ to find

$$\binom{h}{g} s_n(h) \leqslant \left\{ \prod_{k=1}^{g} \left[(n + i_{n_k} - 3) \left(\left\lceil \frac{n + i_{n_k}}{4} \right\rceil - 1 \right) \right] \right\} s_{n+j_n}(h-g)$$

where $-g \leqslant i_{n_k} \leqslant 2g$ and $-g \leqslant j_n \leqslant 2g$, for all n and k.

Proof. Choose g boundary components from h and apply lemma 4.4.

5. Growth constants

In this section we investigate the growth constants of the sets $\mathscr{S}_n(h)$ in n. In § 5.1 we prove that the limits (1.4) exist and that they are all equal (equation (1.5)). In § 5.2 we prove that the growth constant of $\mathscr{S}_n(0)$ is strictly smaller than that of $\mathscr{S}_n(h)$ if $h \geq 1$. Finally, in § 5.3 we prove that in two dimensions $\beta_h < \beta$ as in equation (1.7).

5.1. $\beta_l = \beta_h$

In this subsection, we take the results from concatenation (theorem 3.3) and show that, with the results of the stripping operation, it is possible to prove the existence of the limit (1.4). In $d \geq 3$ dimensions, we prove the existence of β_1 before we look at the more general case.

Theorem 5.1. Let $d \geq 3$. Then there exists a finite, positive real number β_1, which depends on d, such that

$$\lim_{n\to\infty} n^{-1} \log s_n(1) = \log \beta_1 .$$

Proof. Put $h = g = 1$ in theorem 3.3(iv). Then

$$s_n(1) s_m(1) \leq s_{n+m+10}(2).$$

From lemma 4.4 we have

$$s_n(1) s_m(1) \leq (n+m+7)\left(\left\lceil \frac{n+m+10}{4} \right\rceil - 1\right) s_{n+m+10+i_p}(1)$$

where $p = n+m+10$ and i_p is a number between -1 and 2. By theorem 2.1 $s_n(1)^{1/n}$ is a sequence of numbers bounded above, and the theorem follows from the results of [23-25].

We now prove the main result of this subsection. The proof is slightly different in $d = 2$ and in $d \geq 3$ dimensions, so we shall give a separate proof for each of these cases.

Theorem 5.2. Let $d \geq 2$. Then there exist finite, positive real numbers $\beta_h \forall h \geq 1$, which depend on d, such that $\beta_1 = \beta_h$ for all h and

(i) $$\lim_{n\to\infty} n^{-1} \log s_n(h) = \log \beta_h \qquad \text{if } d = 2$$

(ii) $$\lim_{n\to\infty} n^{-1} \log s_n(h) = \log \beta_h \qquad \text{if } d \geq 3.$$

Proof. (i) Let $d = 2$. We suppose that this limit exists for h and prove that it then exists for $h+1$. From theorem 3.3(iii) and lemma 4.2 we find

$$s_{n-m}(h) s_m(2) \leq s_n(h+1) \leq \left\lfloor \frac{n-6}{2} \right\rfloor \left(\left\lceil \tfrac{1}{2}(\sqrt{2n+1} - 1)\right\rceil - 1\right) s_n(h)$$

where m is chosen such that $s_m(2) > 0$ and then fixed. Taking the logarithm, dividing by n and letting n go to infinity, gives

$$\log \beta_h \leq \lim_{n\to\infty} n^{-1} \log s_n(h+1) \leq \log \beta_h$$

so that $\beta_h = \beta_{h+1}$. The second inequality in the statement follows directly from the result in theorem 3.3(i). Since β_1 exists, β_h exists for all h and is equal to β_1.

(ii) Let $d \geqslant 3$. Suppose that the limit exists for h. Since it exists for $h = 1$ we have to prove that it exists for $h+1$. From theorem 3.3(iv) and lemma 4.4 we find

$$s_{n-m}(h) s_{m-1\,0}(1) \leqslant s_n(h+1) \leqslant (n-3)\left(\left\lceil \frac{n}{4} \right\rceil - 1\right) s_{n+i_n}(h)$$

where m is chosen such that $s_{m-1\,0}(1) > 0$ and is then fixed, and i_n is an integer between -1 and 2 inclusive. Taking logarithms, dividing by n and letting n go to infinity gives $\beta_h = \beta_{h+1}$ so that $\beta_1 = \beta_h$ for all $h \geqslant 1$.

5.2. $\beta_0 < \beta_1$ in $d \geqslant 3$ dimensions

At this point we have not derived a relationship between β_0 and β_1 in $d \geqslant 3$ dimensions. Concatenation indicates that $s_n(0) s_m(1) \leqslant s_{n+m+1\,0}(1)$ proving that $\beta_0 \leqslant \beta_1$. Numerical evidence that the inequality is strict can be found in the calculation by Glaus [9]. Here we prove that it is strict by constructing a lower bound on β_1 that is strictly greater than an upper bound on β_0.

Lemma 5.3 (Durhuus et al [16]). Let $d \geqslant 3$. Then

$$s_n(0) \leqslant \binom{d}{2} (2d-3)^{n-1}.$$

Lemma 5.4. Let $d \geqslant 2$. Then

$$s_n(1) \geqslant \binom{d}{2} (2d-2)^{n-1}.$$

Proof. Consider first the proof for $d = 2$. Suppose we deposit a plaquette in the plane. This plaquette has four edges, two of which have endpoints with larger first or second components than the other two. Choose one of these edges and add a second plaquette incident on this edge in a positive lattice direction (e_1 or e_2). Repeat this process with the last plaquette added. Repeating this process n times we note that with each plaquette added we have two possible outcomes, giving 2^{n-1} for n plaquettes.

In $d \geqslant 3$ dimensions the construction is similar. Embed a plaquette in $\mathscr{Z}^d$ in one of $\binom{d}{2}$ orientations. Suppose its edges are in the e_i and e_j directions. Two of the edges of this plaquette are incident on the top vertex of the plaquette. Choose one of these, say e_j, and add a plaquette in the (e_j, e_k) plane to it, $1 \leqslant k \leqslant d$ and $k \neq j$. This plaquette can thus be added in $(d-1)$ possible orientations. Since we are adding on two edges we can add this plaquette in $2(d-1)$ different ways. Repeat this process now $(n-1)$ times, always adding onto the last plaquette added. All the objects created in this way have one boundary component, so that they are all in $\mathscr{S}_n(1)$. This proves the theorem.

Theorem 5.5. Let $d \geqslant 3$. Then

$$\beta_0 < \beta_1.$$

Proof. By lemmas 5.3 and 5.4

$$\lim_{n\to\infty} n^{-1} \log s_n(0) \leqslant \log(2d-3)$$

and

$$\lim_{n\to\infty} n^{-1} \log s_n(1) \geqslant \log(2d-2)$$

so that $\beta_0 < \beta_1$.

5.3. $\beta_l < \beta$ in $d=2$ dimensions

In this subsection we prove the above inequality for two dimensions. To prove this result, we use the strong inequality produced by the 'drilling' construction in two dimensions in § 4. We do not have an equivalent result for $d \geqslant 3$ dimensions, so that we are unable to extend this result to higher dimensions.

Theorem 5.6. Let $d=2$. Then $\beta_1 < \beta$.

Proof. By theorem 3.4(i), for any $\varepsilon > 0$ we can find a $n_0 \in \mathcal{N}$ such that

$$(\beta_1 - \varepsilon)^n \leqslant s_n(1) \leqslant (\beta_1 + \varepsilon)^n$$

for all $n \geqslant n_0$ and where n_0 goes to infinity if $\varepsilon \to 0$. Thus

$$\begin{aligned} s_n &= \sum_{h=1}^{\infty} s_n(h) \\ &\geqslant \sum_{h=0}^{\infty} C^{-h} \binom{\lfloor \frac{n-48h}{49} \rfloor}{h} s_{n-48h}(1) && \text{by lemma 3.1} \\ &\geqslant \sum_{h=0}^{\infty} C^{-h} \binom{\lfloor \frac{n-48h}{49} \rfloor}{h} (\beta_1 - \varepsilon)^{n-48h} && \forall n \geqslant n_0 \end{aligned}$$

The combinatorial factor in the above is 0 unless $h \leqslant \lfloor \frac{n-48h}{49} \rfloor$. Hence, $h \leqslant \frac{n}{97}$, and since $\lfloor \frac{n}{97} \rfloor \leqslant \lfloor \frac{n-48h}{49} \rfloor$ we find

$$\begin{aligned} s_n &\geqslant \sum_{h=0}^{\lfloor \frac{n}{97} \rfloor} C^{-h} \binom{\lfloor \frac{n}{97} \rfloor}{h} (\beta_1 - \varepsilon)^{n-48h} \\ &= (\beta_1 - \varepsilon)^n (1 + C^{-1}(\beta_1 - \varepsilon)^{-48})^{\lfloor \frac{n}{97} \rfloor}. \end{aligned}$$

Take logarithms, divide by n, and let $n_0 \to \infty$. Then $n \to \infty$ and we can take $\varepsilon \to 0^+$, so that

$$\begin{aligned} \beta &= \lim_{n\to\infty} \frac{1}{n} \log s_n \\ &\geqslant \beta_1 + \tfrac{1}{97} \log(1 + C^{-1}\beta_1^{-48}) \\ &> \beta_1. \end{aligned}$$

6. Additional results

There is a large body of numerical evidence that the number of self-avoiding walks with n edges, c_n, has the asymptotic behaviour

$$c_n \sim n^{\gamma-1}\mu^n \tag{6.1}$$

especially from series analysis work [26] and the study of critical phenomena in magnets [27]. μ is the growth constant of the self-avoiding random walk (a lattice-dependent number equivalent to β in this paper) and γ is a critical exponent (usually called the susceptibility exponent through the connection it has in the study of critical phenomena). A rigorous proof of (6.1) is still outstanding, and it is only known rigorously that $c_n = \mu^{n+O(\sqrt{n})}$. The study of bond (edge) animals with fixed cyclomatic index [13] (trees have cyclomatic index zero) leads to the introduction of a set of similar critical exponents for lattice animals. If $a_n(c)$ is the number of animals with cyclomatic index c and n edges, then it is known rigorously that $a_n(c) = \lambda_0^{n+o(n)}$, and postulated that

$$a_n(c) \sim n^{-\theta_c}\lambda_0^n \tag{6.2}$$

where λ_0 is a growth constant and θ_c is an exponent depending on the number of cycles, c, of the animals. For edge animals, Soteros *et al* [13] proved the relationship $\theta_{c+1} = \theta_c - 1$.

In the case of surfaces on the lattice with a fixed number of boundary components we have proven in § 5 that (theorems 3.4, 5.1 and 5.2) $s_n(h) = \beta_h^{n+o(n)}$. It seems reasonable to postulate (in view of the results obtained for walks and, more importantly, for lattice animals) that

$$s_n(h) \sim n^{-\phi_h}\beta_h^n. \tag{6.3}$$

ϕ_h is again a critical exponent associated now with self-avoiding random surfaces. There is some numerical evidence for this assumption, at least for the cases $h = 0$ and 1 [9]. It is now of interest to see what the constructions in this paper imply for the exponent ϕ_h. Immediately we have the following theorem.

Theorem 6.1. Suppose that for every $h \geq 1$ there exists a constant C_h such that

$$s_n(h) \sim C_h n^{-\phi_h}\beta_h^n.$$

Then the exponents ϕ_h are related to each other via the following relations:

(i) $d = 2$;

$$\phi_h \geq \phi_{h+1}$$
$$\phi_1 - h \geq \phi_{h+1} \geq \phi_1 - \tfrac{3}{2}h \qquad \forall h \geq 1$$

(ii) $d \geq 3$;

$$\phi_h \geq \phi_{h+1} \geq \phi_1 - 2h \qquad \forall h \geq 1.$$

Proof. (i) The first inequality follows directly from the results of concatenation. Consider theorem 3.3(iii), let $g = 2$; then

$$s_n(h)s_m(2) \leq s_{n+m}(h+1).$$

Fix $m>0$ at a value such that $s_m(2)>0$. Substitute the assumption, divide by β_1^n, take logarithms, divide by $\log n$ and let $n\to\infty$. This produces $-\phi_h\leqslant-\phi_{h+1}$.

The second inequality follows from lemma 4.1 and corollary 4.3:

$$C^{-h}\binom{\lfloor\frac{n-48h}{49}\rfloor}{h}s_{n-48h}(1)\leqslant s_n(h+1)\leqslant(h+1)^{-1}\left(\left\lfloor\frac{n-6}{2}\right\rfloor(\lceil\tfrac{1}{2}(\sqrt{2n+1}-1)\rceil-1)\right)^h s_n(1).$$

Performing the series of operations above and using $\binom{an}{b}\sim(an)^b$ produces

$$-\phi_1+h\leqslant-\phi_{h+1}\leqslant-\phi_1+\tfrac{3}{2}h.$$

(ii) This result can be directly derived from theorem 3.3 (iv) and corollary 4.5:

$$s_n(h)s_{m-10}(1)\leqslant s_n(h+1)\leqslant(h+1)^{-1}\left\{\prod_{k=1}^{h}(n+j_{n_k}-3)\left(\left\lceil\frac{n+j_{n_k}}{4}\right\rceil-1\right)\right\}s_{n+i_n}(1)$$

where m is a number chosen such that $s_{m-10}(1)>0$ and $-h\leqslant i_n, j_{n_k},\leqslant 2h$ for all n and k. The numbers i_n and j_{n_k} are bounded for each n and fixed value of h, the number of boundary components. Performing the same series of manipulations as above gives the desired results.

7. Conclusions

In this paper we classified self-avoiding surfaces by the numbers of distinct boundary components, and devised constructions that would either increase (drilling) or decrease (stripping) the number of boundary components of a surface in analogy with the cyclomatic index defined for lattice animals [13, 14]. These operations were combined to prove the existence of growth constants β_h on the hypercubic lattice, and gave the bounds in § 6 on the critical exponent ϕ_h. The following observations can now be made.

(1) Apart from proving the existence of the growth constants β_h for the sets $\mathcal{S}_n(h)$ we also proved in § 5.1 that the growth constants are independent of the number of boundary components of a set, that is $\beta_1=\beta_h$ for all $h\geqslant 1$ in all dimensions $d\geqslant 2$. In $d\geqslant 3$ dimensions we also proved that the growth constant of $\mathcal{S}_n(0)$ is strictly smaller than β_1 and in $d=2$ dimensions we proved that $\beta_1<\beta$, where β is the growth constant of the set $\mathcal{S}_n$. We are at present unable to extend this result to $d\geqslant 3$ dimensions, primarily because we cannot prove lemma 4.1 in higher dimensions.

(2) In this paper we have considered self-avoiding surfaces with any genus and with boundary components which can be knotted or linked. Alternatively, we could consider surfaces with the topology of a punctured sphere (with zero genus) and in which the boundary components are unknotted and unlinked. In this case classifying by the number of boundary components is equivalent to classifying by the rank of the first homology group as seen from equation (1.1). Our arguments apply equally to this case and establish the existence of growth constants β_h^0. We can show that $\beta_h^0=\beta_1^0\forall h\geqslant 1$ and $\beta_0^0<\beta_1^0$ for $d\geqslant 3$. This case has been examined numerically by Glaus [9] who found

$$\beta_0^0=1.733\pm0.006 \tag{7.1}$$

$$\beta_1^0=12.798\pm0.018 \tag{7.2}$$

in three dimensions.

(3) Similarly we could confine our attention to orientable surfaces with h boundary components. The same arguments apply and we find similar results.

(4) The critical exponents in § 6 were bounded by applying the lemmas derived in §§ 3 and 4. It is now known that for lattice animals, the exponents θ_c satisfy the relationship

$$\theta_c = \theta_0 - c. \tag{7.3}$$

Here the index c is the cyclomatic index of the animals. The bounds derived in theorem 6.1 on the exponents ϕ_h for surfaces are consistent with a similar relation for self-avoiding surfaces, and we conjecture that

$$\phi_{h+1} = \phi_1 - h \tag{7.4}$$

in $d \geq 2$ dimensions. The exponent ϕ_1 is believed to be that of branched polymers [9, 10, 28, 29], and was found by Glaus to be

$$\phi_1 = 1.48 \pm 0.05 \tag{7.5}$$

in three dimensions.

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